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Left Eigenstructure Assignment via Sylvester Equation

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An effective and disturbance suppressible controller can be designed by assigning a left eigenstructure (eigenvalues/left eigenvectors) of a system. In this note, a novel left eigenstructure assignment scheme via Sylvester equation is proposed. The biorthogonality property between the right and left modal matrices of a system is utilized to develop the scheme.

Key Words: Left Eigenstructure, Sylvester Equation, Biorthogonality Condition

1. Introduction

Eigenstructure (eigenvalues/eigenvectors) assignment via linear state feedback control in a linear multivariable system has been widely used as a control scheme in mode decoupling of the flight control system and vibration suppression of flexible structures.

The specified effect of the controller is achieved by assigning a certain set of eigenvalues and an associated set of eigenvectors to the closed-loop system. The eigenstructure assignment algorithm can be divided into two groups; namely, the right eigenstructure (eigenvalues/right eigenvectors) assignment and the left eigenstructure (eigenvalues/left eigenvectors) assignment. Their roles in designing a control system are distinctly different.

Consider a linear time invariant multivariable controllable system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ef(t) = Ax(t) + \sum_{k=1}^{m} b_k u_k(t) + \sum_{l=1}^{n} e_l f_l(t), \quad (1)$$

$$u(t) = Kx(t) \quad (2)$$

where $(i)\, x \in R^N$, $u \in R^m$, and $f \in R^n$ denote the state, control, and disturbance vectors, respectively; (ii) A, B, E, and K are real constant matrices of appropriate dimensions; and (iii) rank $B = m \neq 0$.

The response of the given system due to control input $u(t)$ and disturbance $f(t)$ with zero initial conditions is represented using the modal matrices by

$$x(t) = \Phi \int_0^t e^{\Lambda(t-\tau)} \{ \Psi^T B u(\tau) + \Psi^T E f(\tau) \} d\tau = \sum_{i=1}^{N} \phi_i e^{\lambda_i t} \left\{ \sum_{k=1}^{m} (\psi_i^T b_k) \int_0^t e^{-\lambda_i \tau} u_k(\tau)\, d\tau + \sum_{l=1}^{n} (\psi_i^T e_l) \int_0^t e^{-\lambda_i \tau} f_l(\tau)\, d\tau \right\} \quad (3)$$

where Φ and Ψ are the right and left modal matrices of the given system, respectively, and Λ is the diagonalized eigenvalues matrix. Note, from Eq. (3), that the response to disturbance can be eliminated if the columns (ψ_i) of left modal matrix Ψ are orthogonal to the columns (e_l) of the disturbance input matrix E. Note also that the control effort is effectively transferred (that is, the manipulation is achieved with small control effort), if the left eigenvectors are parallel to the columns (b_k) of the control input matrix B. Therefore, for both effective control and disturbance suppression, it is desired that the left eigenvectors of the system lie simultaneously (at least, in the least square sense, if possible) in the space orthogonal to the columns of the disturbance input matrix E and parallel to the columns of the control input matrix B. Then, the corresponding system can be manipulated with small effort without being disturbed by the disturbance

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input. Therefore, it can be said that the left eigenstructure of a system plays an important role in designing an effective and disturbance suppressible controller.

From the above equations, the right eigenstructure (λ_i, ϕ_i) assignment is used to solve mode decoupling problems (Andry, Shapiro, and Chung, 1983; Kang and Lee, 1992; Sobel and Cloutier, 1992; Siouris, Lee, and Choi, 1995; Choi *et al.,* 1996), and to design a controller for the vibration suppression of flexible structures (Garrard and Liebst, 1985; Song and Jayasuriya, 1993). On the other hand, control effectiveness and disturbance suppressibility of a system depend mainly on the left eigenstructure (λ_i, ψ_i) of the system (Zhang, Slater, and Allemang, 1990; Kim and Junkins, 1991; Choi *et al.,* 1993a, 1993b, 1994, 1995). Zhang et al. (1990) used a left eigenstructure to suppress undesired inputs, through orthogonalizing left eigenvectors to disturbance input matrix of the system of uniform flexible beam vibration control problem. Kim and Junkins (1991) utilized the left eigenstructure to improve the controllability of a flexible structure system through placing actuators at optimal locations. However, Zhang *et al.* did not take into account the control problems; and Kim and Junkins did not consider the disturbance suppression problems. Choi *et al.* (1993a, 1993b, 1994, 1995) proposed a left eigenstructure assignment method which considers the two problems simultaneously by using a null space approach (Choi *et al.,* 1995). Recently, Choi (1998) also proposed a novel simultaneous assignment methodology of right and left eigenstructures via the null space approach. In the paper, the method was successfully applied to the design of a stability augmentation system (SAS) with closure of the roll attitude loop for the linearized lateral dynamics of an L-1011 aircraft.

An algorithm for state feedback pole assignment using Sylvester equation was introduced in (Bhattacharyya and deSouza, 1982) and used in (Cavin III and Bhattacharyya, 1983) to have low eigenvalue sensitivity for the closed-loop system, and the problem of right eigenstructure assignment via Sylvester equation has been treated by several authors (Keel and Bhattacharyya, 1985; Tsui, 1987; Duan, 1993; Kim and Kum, 1993; Syrmos and Lewis, 1993, 1994; Wimmer, 1994). Keel and Bhattacharyya (1985) described a procedure for the design of a dynamic compensator that stabilizes the closed-loop system and causes the closed-loop system eigenstructure to be robust in the sense of making the eigenvector set maximally orthonormal. The authors extended the algorithm introduced in Refs. (Bhattacharyya and deSouza, 1982; Cavin III and Bhattacharyya, 1983), to the output feedback case and place eigenvalues in a region of the complex plane. Tsui (1987) summarized the existing solutions to Sylvester equation, and also presented an attractive analytical and restriction-free solution with explicit freedom. Duan (1993) proposed two new simpler solutions to Sylvester equation than that of Tsui, and presented a complete parametric approach for right eigenstructure assignment in linear systems via state feedback based on his proposed solutions. Syrmos and Lewis (1993) solved the problem of eigenstructure assignment by output feedback by using two coupled Sylvester equations. In Kim and Kum (1993), the authors introduced an iterative right eigenstructure assignment via Sylvester equation to design a small gain controller. A homotopy concept was adopted to develop the scheme. Syrmos and Lewis (1994) also presented necessary and sufficient conditions in terms of a bilinear Sylvester equation for stabilizing and eigenstructure assignment by output feedback.

However, the problem of left eigenstructure assignment via Sylvester equation could not be solved directly because of the inherent structure of the Sylvester equation. In this note, a novel left eigenstructure assignment scheme via Sylvester equation is proposed. The whole procedure is attractively simple to use in designing a left eigenstructure of a system. The simplicity and usefulness of the presented method is illustrated by a numerical example.

2. Sylvester Equation

Consider Eq. (1) in section 1. If a constant real state feedback (Eq. (2)) is applied to Eq. (1), the closed-loop system becomes

$$\dot{x}(t) = (A+BK)x(t) + Ef(t) \quad (4)$$

and corresponding right and left eigenvalue problems are defined by

$$(A+BK)\phi_i = \lambda_i\phi_i : \text{right} \quad (5)$$
$$(A+BK)^T\psi_i = \lambda_i\psi_i : \text{left} \quad (6)$$

where ϕ_i and ψ_i are the right and left eigenvectors, respectively, corresponding to the eigenvalue λ_i.

The central constraint imposed in the eigenvalue assignment problem is to determine the gain matrix K that results in a prescribed set of eigenvalues. Note that K is an $m \times N$ dimensional matrix; it is evident that the problem is underdetermined, and therefore for controllable systems, an infinity choices of gain matrices exist for given eigenvalue locations. We could choose $N \times (m-1)$ parameters arbitrarily for N prescribed eigenvalues.

The pole placement algorithm proposed in Cavin III and Bhattacharyya (1983) introduces the parameter vector $h_i \in C^m$ defined by

$$h_i = K\phi_i \quad (7)$$

Then Eq. (5) is put in the form of Sylvester equation:

$$(A - \lambda_i I)\phi_i = -Bh_i \quad (8)$$

or, in matrix form, Eq. (8) is a generalized Lyapunov equation known as Sylvester equation,

$$A\Phi - \Phi\Lambda = -BH \quad (9)$$

where $\Phi = [\phi_1, \phi_2, \cdots, \phi_N]$, $\Lambda = \text{diag}\ [\lambda_1, \lambda_2, \cdots, \lambda_N]$, and $H = [h_1, h_2, \cdots, h_N]$.

The pole placement scheme based on Sylvester equation (Eq. (8) or (9)) can be summarized as follows: For given set of A, B matrices, and for a prescribed Λ matrix, we can choose a parameter matrix H and solve for Φ from Eq. (9); Then, we can solve for K from the linear system (which is simply the matrix version of Eq. (7)).

$$K\Phi = H. \quad (10)$$

In essence, the advantage of "guessing Λ and H" instead of "guessing K" is that the exact prescribed eigenvalue positions are guaranteed if we specify Λ and choose appropriate H. The H matrix generates (through the solution of Eq. (9) for Λ specified) all infinity of achievable eigenvector matrices (Junkins and Kim, 1993).

Note, from inversion of Eq. (8), that the closed-loop eigenvectors corresponding to given λ_i and h_i are simply

$$\phi_i = -(A - \lambda_i I)^{-1}Bh_i \quad (11)$$

Thus, if the closed-loop eigenvalues (λ_i) are distinct from their open-loop positions, the columns of H directly generate all possible corresponding closed-loop eigenvectors.

In case of right eigenstructure assignment, unfortunately, an arbitrary choice for the complex H matrix does not usually generate an attractive set of closed-loop eigenvectors; occasionally the resulting eigenvectors are so poorly conditioned that computing an accurate gain matrix K from Eq. (10) is not possible.

Since an arbitrary selection of H is not appropriate, we had better consider choices which have a high probability of generating attractive gain matrices. An attractive algorithm results if we seek the H matrix which makes the closed-loop modal matrix lie as close as possible to a prescribed, well-conditioned matrix. Notice that, if we select some target set of well-conditioned closed-loop eigenvectors

$$\hat{\Phi} = [\hat{\phi}_1, \hat{\phi}_2, \cdots, \hat{\phi}_N] \quad (12)$$

which is much easier than choosing H matrix, considering physical information of Φ, then, we can use Eq. (8), or equivalently Eq. (9), to solve for the $\hat{H}$ that most nearly (*e. g.,* in the least square sense) produces the desired eigenvectors $\hat{\Phi}$. Upon substituting this solution for the H matrix and re-solving Eq. (9) for the admissible eigenvector matrix Φ, we will find $\Phi \neq \hat{\Phi}$, exactly, with the degree of approximation being problem-dependent. The resulting Φ matrix lies as near $\hat{\Phi}$ as possible (*i. e.,* in the least square sense) and is typically well conditioned. The gain matrix K calculated from solution of Eq. (10) with Φ and $\hat{H}$ will, however, place the eigenvalues exactly, to within arithmetic errors.

3. Left Eigenstructure Assignment via Sylvester Equation

Consider the left eigenvalue problem of Eq. (6). The equation can be expressed by the following Sylvester equation:

$$A^T \Psi_d - \Psi_d \Lambda = -K^T B^T \Psi_d. \tag{13}$$

Our objective is to find the feedback gain matrix K that the closed-loop eigenvalues are achieved exactly and the desired left eigenvectors are assigned to the best possible set of eigenvectors, at least, in the least square sense. However, the matrix K and the desired nonsingular left modal matrix Ψ_d in the right hand side of Eq. (13) can not be parameterized by another matrix such that H in Eq. (9) because of the inherent structure of the right hand side of Eq. (13). Hence, the left eigenstructure can not be assigned by direct application of Eq. (13).

Now, we reformulate Eq. (13) to be parameterized by the following procedures, and then the reformulated equation can be used to a left eigenstructure assignment problem. Eq. (13) can be rewritten by postmultiplying Ψ_d^{-1} in both sides as follows:

$$A^T - \Psi_d \Lambda \Psi_d^{-1} = -K^T B^T \tag{14}$$

Premultiply Eq. (14) by Ψ_d^{-1} and then take transpose in both sides to get

$$A \Psi_d^{-T} - \Psi_d^{-T} \Lambda = -BK\Psi_d^{-T}. \tag{15}$$

Then, the matrices K and $\Psi_d^{-T} (\equiv \Gamma)$ in the right hand side of Eq. (15) can be parameterized by

$$p_i = K\gamma_i, \tag{16}$$

and Roman (15) can be rewritten by

$$\mathrm{A}\Gamma = \Gamma\Lambda = -BP \tag{17}$$

where γ_i is the i-th column vector of the matrix Γ, and $P = [p_1, p_2, \cdots, p_N]$.

Note that the right modal matrix Φ in Eq. (9) is replaced by Γ(or Ψ_d^{-T}) in Eq. (17) as a target left modal matrix. A left modal matrix of a system can be assigned to the desired one, at least, in the least square sense, guaranteeing the desired closed-loop eigenvalues to be achieved exactly using the conventional Sylvester equation after some matrix manipulations.

From the above facts, we obtain the following algorithm for a left eigenstructure assignment of a given controllable system.

Algorithm:

- *Step 1:* Choose the diagonalized desired closed-loop spectrum Λ and the desired nonsingular left modal matrix Ψ_d.
- *Step 2:* Calculate the matrix $\Gamma (= \Psi_d^{-T})$ and take this matrix as a target left modal matrix.
- *Step 3:* Calculate the parameter matrix P as follows:

$$P = -B\dagger(A\Gamma - \Gamma\Lambda).$$

where $B\dagger$ denotes the pseudo-inverse of matrix B.

- *Step 4:* Solve the Sylvester equation (Eq. (17)) for the left eigenstructure assignment problem with the matrix P calculated in *Step 3* to get the achievable matrix Γ_a. That is, we solve the following Sylvester equation for Γ_a:

$$A\Gamma_a - \Gamma_a\Lambda = -BP.$$

- *Step 5:* Calculate the feedback gain matrix as follows:

$$K = P\Gamma_a^{-1}.$$

- *Step 6:* Calculate the achievable left modal matrix Ψ_a as follows:

$$\Psi_a = \Gamma_a^{-T}.$$

Remarks:

1) In *Step 1* of the algorithm, the desired left modal matrix Ψ_d can be determined to have the specified control effectiveness and disturbance suppressibility in order to obtain an effective and disturbance suppressible controller. A procedure to determine the desired left modal matrix Ψ_d is given in Choi *et al.*, (1995). The characteristics, control effectiveness and disturbance suppressibility, of the system are determined by the direction of each column vector of the designed left modal matrix Ψ_d.

2) In *Step 2*, if the desired left modal matrix Ψ_d is close to singular, the algorithm using singular value decomposition (Junkins and Kim, 1993) is recommended to improve the accuracy when

calculating the pseudo-inverse of the matrix.

3) In the algorithm, if the rank of control input matrix B (that is, the number of independent control actuators) is increased up to the rank of system matrix A, the desired left eigenvectors as well as the desired eigenvalues are exactly achieved.

4. A Numerical Example

Consider a third-order two-input continuous controllable linear system with a disturbance:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ef(t)$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f(t).$$

Let the desired real distinct spectrum be $\Lambda = \text{diag}[-2, -3, -4]$, and the desired left modal matrix Ψ_d be

$$\Psi_d = \begin{bmatrix} -6 & 3 & -3 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

where Ψ_d is determined according to the guideline described in Choi *et al.*, (1995) to have the specified control effectiveness and disturbance suppressibility. Normalized matrix Ψ_d^{nor} using $(\psi_i^T \psi_j = \delta_{ij})$

$$\Psi_d^{\text{nor}} = \begin{bmatrix} -0.9864 & 0.9487 & -0.9045 \\ -0.1644 & 0.3162 & 0.3015 \\ 0 & 0 & 0.3015 \end{bmatrix}.$$

Then, the matrix Γ is obtained by

$$\Gamma = \begin{bmatrix} -0.3333 & -0.3333 & 0 \\ 1 & 2 & 0 \\ -2 & -3 & 1 \end{bmatrix},$$

and the parameter matrix P is obtained in the least square sense by

$$P = \begin{bmatrix} 6.3333 & 12.3333 & -6 \\ 0 & -3 & -1 \end{bmatrix}.$$

Now, we obtain the matrix Γ_a by solving the Sylvester equation in *Step 4* as follows:

$$\Gamma_a = \begin{bmatrix} -0.5278 & -0.6833 & 0 \\ 1.0566 & 2.0500 & 0 \\ -2.111 & -3.1500 & 1 \end{bmatrix}.$$

Then, the feedback gain matrix K is calculated using the obtained matrices P and Γ_a by

$$K = \begin{bmatrix} 16.7805 & 2.3902 & -6 \\ -6 & -5 & -1 \end{bmatrix},$$

and therefore the closed-loop system

$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ -6 & -5 & 0 \\ 14.7805 & 3.3902 & -4 \end{bmatrix}$$

which has the exact desired spectrum Λ. The normalized achievable left modal matrix Ψ_a^{nor} is obtained by

$$\Psi_a^{\text{nor}} = \begin{bmatrix} -0.9487 & 0.8944 & -0.9216 \\ -0.3162 & 0.4472 & 0.2021 \\ 0 & 0 & 0 \end{bmatrix},$$

which satisfies the normalized desired left modal matrix Ψ_d^{nor} in the least square sense.

5. Conclusions

An effective and disturbance suppressible controller can be designed by assigning a left eigenstructure of a system. However, the problem of left eigenstructure assignment via Sylvester equation has not been solved directly because of the inherent structure of the Sylvester equation. In this note, a left eigenstructure assignment scheme via Sylvester equation has been proposed. The proposed left eigenstructure assignment scheme via Sylvester equation guarantees that the desired eigenvalues are achieved exactly and the desired left eigenvectors are assigned to the best possible (achievable) set of eigenvectors, at least, in the least square sense. Thus, the proposed scheme could be utilized in designing a disturbance suppressible as well as an effective controller because the directions of the assigned left eigenvectors govern the degrees of the control effectiveness and disturbance suppressibility of the system. A numerical example has confirmed the simplicity and usefulness of the proposed left eigenstructure assignment scheme via Sylvester equation.

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